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Research Article

Mathematical Analysis of Visceral Leishmaniasis Model

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Abstract. In this work, we consider a mathematical model describing the dynamics of visceral leishmaniasis in a population of dogs D. First, we consider the case of constant total population D, this is the case where birth and death rates are equal, in this case transcritical bifurcation occurs when the basic reproduction number $\mathscr{R}_0$ is equal to one, and global stability is shown by the mean of suitable Lyapunov functions. After that, we consider the case where the birth and death rates are different, if the birth rate is great than death rate the total dog population increases exponentially, while the infectious dogs I dies out if the basic reproduction number is less than one, if it is great than one then D goes to infinity. We also prove that the total population D will extinct for birth rate less than death rate. Finally we give numerical simulations.

Keywords: Visceral leishmaniasis model, Bifurcation, Global stability, Lyapunov functions, asymptotic properties.

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1. Introduction

In this paper, we investigate a mathematical model of zoonotic visceral leishmaniasis (ZVL), the model studied here is inspired from [1, 4, 5]. Zoonotic visceral leishmaniasis (ZVL) caused by Leishmania infantum is a disease of humans and domestic dogs (the reservoir) transmitted by phlebotomize sandflies. According to the World Health Organization, leishmaniasis is one of the diseases affecting the poorest in developing countries, 350 million people are considered at risk of contracting leishmaniasis (see [11]).

Many works have considered mathematical models for ZVL, we can cite [1, 4–10], where behavior of infection, and stability of free disease and endemic equilibria are studied.

Following [1, 4], at time t let the dog population of size $D(t)$, it is divided into two categories, ever-infectious dogs (that become infectious) and never-infected dogs. Ever-infectious dogs category is partitioned into three subclasses who are susceptible (uninfected), latent (infected but not infectious) and infectious dogs, with sizes (numbers) denoted by $S(t)$, $L(t)$ and $I(t)$ respectively. Never-infected dogs category is partitioned into two subclasses who are uninfected and infected dogs, with sizes denoted by $R(t)$ and $Q(t)$ respectively (Figure 1).

The sum $S(t) + L(t) + I(t) + R(t) + Q(t)$ is the total population $D(t)$. The natural death rate δ is assumed to be identical in all subclasses. A proportion α of the dogs born susceptible to ZVL with $0 < \alpha < 1$. Consequently, the birth flux into the susceptible class is $\alpha\beta D(t)$ and into the resistant class is $(1 - \alpha)\beta D(t)$ where β is the natural birth rate of dogs. The latent dogs become infectious and re-enter into infected class with rate σ. The force of infection is

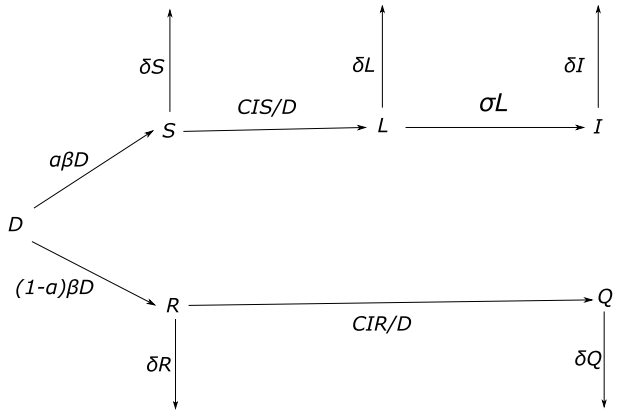

Figure 1: Compartmental model.

$CI(t)/D(t)$, where C is the vectorial capacity of the sandfly population transmitting infection between dogs. It is denoted by $CI(t)S(t)/D(t)$ (resp. $CI(t)R(t)/D(t)$) for the contact between infectious and susceptible (resp. infectious and uninfected) dogs.

Based on the above assumptions, we obtain the epidemic model governed by the following system of ordinary differential equations (see [1, 4])

$$\begin{cases} \dfrac{dS}{dt} = \alpha\beta D - \dfrac{CIS}{D} - \delta S, \\ \dfrac{dL}{dt} = \dfrac{CIS}{D} - (\sigma + \delta)\, L, \\ \dfrac{dI}{dt} = \sigma L - \delta I, \\ \dfrac{dR}{dt} = (1-\alpha)\,\beta D - \dfrac{CIR}{D} - \delta R, \\ \dfrac{dQ}{dt} = \dfrac{CIR}{D} - \delta Q \end{cases} \tag{1}$$

with initial conditions

$$S(0) \geq 0, L(0) \geq 0, I(0) \geq 0, R(0) \geq 0 \text{ and } Q(0) \geq 0. \tag{2}$$

From system (1), it follows that the total dog population $D(t) = S(t) + L(t) + I(t) + R(t) + Q(t)$ can be determined from the differential equation $\frac{dD}{dt} = (\beta - \delta)D$ which gives $D(t) = D_0 e^{(\beta-\delta)t}$, where $D_0 = D(0)$.

Therefore

$$\lim_{t\to\infty} D(t) = \begin{cases} 0 & \text{if } \beta < \delta, \\ D_0 & \text{if } \beta = \delta, \\ \infty & \text{if } \beta > \delta. \end{cases}$$

In [1], the case of constant total population is considered (*i.e.* $\beta = \delta$), where well-posedness of (1), (2) is proved and local stability of equilibria is investigated. In fact, the disease free equilibrium $E_f = \left(S^0, L^0, I^0, R^0, Q^0\right) = (\alpha D, 0, 0, (1-\alpha)D, 0)$ is locally asymptotically stable for $\mathscr{R}_0(\delta) = \frac{C\alpha\sigma}{\delta(\sigma+\delta)} < 1$, and unstable for $\mathscr{R}_0(\delta) > 1$. For $\mathscr{R}_0(\delta) > 1$, the endemic equilibrium

$E^* = \left(S^*, L^*, I^*, R^*, Q^*\right) = \left(\frac{\alpha D}{\mathcal{R}_0(\delta)}, \frac{\delta^2 D(\mathcal{R}_0(\delta)-1)}{\sigma C}, \frac{\delta D(\mathcal{R}_0(\delta)-1)}{C}, \frac{(1-\alpha)D}{\mathcal{R}_0(\delta)}, \frac{(1-\alpha)D(\mathcal{R}_0(\delta)-1)}{\mathcal{R}_0(\delta)}\right)$ exists and is locally asymptotically stable.

In this paper, we study the global stability of equilibria of (1) by constructing a suitable Lyapunov functions and using LaSalle's invariance principle when $\delta = \beta$. A critical case $\mathcal{R}_0(\delta) = 1$ is also investigated. After that, we consider the case where $\beta \neq \delta$. Numerical simulations are given in section four to illustrate our results. We end our work by some conclusions.

2. Constant Total Population (Case $\delta = \beta$)

The results of Boukhalfa *et al.* [1] are given in Theorems 2.1 and 2.2.

Theorem 2.1 (see [1]). *Assume that $\beta = \delta$. System (1) has the following equilibria:*

- *The disease free equilibrium $E_f = \left(S^0, L^0, I^0, R^0, Q^0\right) = (\alpha D, 0, 0, (1-\alpha)D, 0)$ which exists always.*
- *If $\mathcal{R}_0(\delta) > 1$, the system (1) admits a unique positive equilibrium $E^* = (S^*, L^*, I^*, R^*, Q^*) = \left(\frac{\alpha D}{\mathcal{R}_0}, \frac{\delta^2 D(\mathcal{R}_0-1)}{\sigma C}, \frac{\delta D(\mathcal{R}_0-1)}{C}, \frac{(1-\alpha)D}{\mathcal{R}_0}, \frac{(1-\alpha)D(\mathcal{R}_0-1)}{\mathcal{R}_0}\right)$ namely the endemic equilibrium.*

Theorem 2.2 (see [1]). *Let $\beta = \delta$.*

- *If $\mathcal{R}_0(\delta) < 1$, then the disease free equilibrium (DFE) E_f of (1) is locally asymptotically stable. If $\mathcal{R}_0(\delta) > 1$, E_f is unstable.*
- *If $\mathcal{R}_0(\delta) > 1$, the endemic equilibrium E^* of (1) is locally asymptotically stable.*

The first four equations of (1) are independent of Q, therefore the last equation of (1) can be omitted without loss of generality. Hence, system (1) is reduced to

$$\begin{cases} \frac{dS}{dt} &= \alpha\beta D - \frac{CIS}{D} - \delta S, \\ \frac{dL}{dt} &= \frac{CIS}{D} - (\sigma + \delta) L, \\ \frac{dI}{dt} &= \sigma L - \delta I, \\ \frac{dR}{dt} &= (1-\alpha)\beta D - \frac{CIR}{D} - \delta R \end{cases} \tag{3}$$

with initial conditions

$$S(0) \geq 0, \ L(0) \geq 0, \ I(0) \geq 0, \ R(0) \geq 0 \tag{4}$$

and $D(t) = D_0 e^{(\beta-\delta)t}$.

The considered region for (3) is

$$\widetilde{\Gamma}_D := \{(S, L, I, R) \in \mathbb{R}^4_+ \backslash S + L + I + R \leq D\}.$$

which is positively invariant since all solutions of (3) in $\widetilde{\Gamma}_D$ remain there for all $t \geq 0$.

Let $\widetilde{E}_f = (S^0, 0, 0, R^0)$ and $\widetilde{E^*} = (S^*, L^*, I^*, R^*)$ be the equilibria of (3).

2.1. Bifurcation analysis

The case $\mathscr{R}_0(\delta) = 1$ corresponding to $C = C_1 = \frac{\delta(\sigma+\delta)}{\alpha\sigma}$.

To prove bifurcation for $\mathscr{R}_0(\delta) = 1$, we use center manifold method described in Castillo-Chavez and Song [2]. To this aim, let $x = (x_1, x_2, x_3, x_4)$ such that $x_1 = S$, $x_2 = L$, $x_3 = I$, $x_4 = R$.

From (3) we obtain

$$\begin{cases} \dfrac{dx_1}{dt} = \alpha\beta D - \dfrac{Cx_3x_1}{D} - \beta x_1 = f_1(x, C), \\ \dfrac{dx_2}{dt} = \dfrac{Cx_3x_1}{D} - (\sigma+\beta)\, x_2 = f_2(x, C), \\ \dfrac{dx_3}{dt} = \sigma x_2 - \beta x_3 = f_3(x, C), \\ \dfrac{dx_4}{dt} = (1-\alpha)\,\beta D - \dfrac{Cx_3x_4}{D} - \beta x_4 = f_4(x, C). \end{cases} \tag{5}$$

The linearization matrix of system (3) around the disease-free equilibrium for $C = C_1$ is

$$A = \begin{pmatrix} -\delta & 0 & -C_1\alpha & 0 \\ 0 & -(\sigma+\delta) & C_1\alpha & 0 \\ 0 & \sigma & -\delta & 0 \\ 0 & 0 & -(1-\alpha)C_1 & -\delta \end{pmatrix}.$$

The eigenvalues of A are $\lambda_1 = -\delta$ with multiplicity two, $\lambda_2 = 0$ and $\lambda_3 = -\sigma - 2\delta$. Zero is a simple eigenvalue of A and the other eigenvalues have negative real parts.

A right eigenvector $W = (w_1, w_2, w_3, w_4)^T$ such that $AW = \lambda_2 W$ (corresponding to the zero eigenvalue) is $W = (-\frac{\sigma+\delta}{\delta}, 1, \frac{\sigma}{\delta}, -(1-\alpha)\frac{C_1\sigma}{\delta^2})^T$ and the left eigenvector $V = (v_1, v_2, v_3, v_4)$ such that $VA = \lambda_2 V$ and $V.W = 1$ is $V = (0, \frac{\delta}{\sigma+2\delta}, \frac{\delta(\sigma+\delta)}{\sigma(\sigma+2\delta)}, 0)$.

Let

$$a = \sum_{k,i,j=1}^{4} v_k w_i w_j \frac{\partial^2 f_k}{\partial x_i \partial x_j}(\alpha D, 0, 0, (1-\alpha)D, C_1)$$

and

$$b = \sum_{k,i=1}^{4} v_k w_i \frac{\partial^2 f_k}{\partial x_i \partial C}(\alpha D, 0, 0, (1-\alpha)D, C_1).$$

The second partial derivatives of f_2 and f_3 are given by

$$\frac{\partial^2 f_2}{\partial x_1 \partial x_3} = \frac{C}{D}, \quad \frac{\partial^2 f_2}{\partial x_1^2} = \frac{\partial^2 f_2}{\partial x_1 \partial x_2} = \frac{\partial^2 f_2}{\partial x_1 \partial x_4} = 0,$$

$$\frac{\partial^2 f_2}{\partial x_1 \partial C} = \frac{x_3}{D}, \quad \frac{\partial^2 f_2}{\partial x_2^2} = \frac{\partial^2 f_2}{\partial x_1 \partial x_2} = \frac{\partial^2 f_2}{\partial x_2 \partial x_3} = \frac{\partial^2 f_2}{\partial x_2 \partial x_4} = 0,$$

$$\frac{\partial^2 f_2}{\partial x_3 \partial x_3} = \frac{\partial^2 f_2}{\partial x_3^2} = \frac{\partial^2 f_2}{\partial x_3 \partial x_4} = 0, \quad \frac{\partial^2 f_2}{\partial x_3 \partial C} = \frac{x_1}{D},$$

$$\frac{\partial^2 f_2}{\partial x_4 \partial x_1} = \frac{\partial^2 f_2}{\partial x_4 \partial x_2} = \frac{\partial^2 f_2}{\partial x_4 \partial x_3} = \frac{\partial^2 f_2}{\partial x_4^2} = 0,$$

$$\frac{\partial^2 f_3}{\partial x_i \partial x_j} = 0, \ 1 \le i, j \le 4.$$

It follows that

$$\begin{aligned} a &= v_2 \sum_{i,j=1}^{4} w_i w_j \frac{\partial^2 f_2}{\partial x_i \partial x_j}(\alpha D, 0, 0, (1-\alpha)D, C_1) \\ &\quad + v_3 \textstyle\sum_{i,j=1}^{4} w_i w_j \dfrac{\partial^2 f_3}{\partial x_i \partial x_j}(\alpha D, 0, 0, (1-\alpha)D, C_1) \\ &= 2 v_2 w_1 w_3 \frac{\partial^2 f_2}{\partial x_1 \partial x_3}(\alpha D, 0, 0, (1-\alpha)D, C_1) \\ &= -2 \frac{\sigma(\sigma+\delta)}{\delta(\sigma+2\delta)} < 0, \end{aligned}$$

and

$$\begin{aligned} b &= v_2 \sum_{i=1}^{4} w_i \frac{\partial^2 f_2}{\partial x_i \partial C}(\alpha D, 0, 0, (1-\alpha)D, C_1) \\ &\quad + v_3 \textstyle\sum_{i=1}^{4} w_i \dfrac{\partial^2 f_3}{\partial x_i \partial C}(\alpha D, 0, 0, (1-\alpha)D, C_1) \\ &= v_2 w_3 \frac{\partial^2 f_2}{\partial x_3 \partial C}(\alpha D, 0, 0, (1-\alpha)D, C_1) \\ &= \frac{\alpha\sigma}{2\delta+\sigma} > 0. \end{aligned}$$

Thus $a < 0$ and $b > 0$, from result of Theorem 4.1 in Castillo-Chavez and Song [2] we deduce the following theorem.

Theorem 2.3. *A transcritical bifurcation occurs at $\mathcal{R}_0(\delta) = 1$. When C crosses C_1, a disease free equilibrium $\widetilde{E}_f$ changes its stability from stable to unstable. Correspondingly, a negative unstable equilibrium $\widetilde{E}^*$ for $C < C_1$, becomes positive and locally asymptotically stable for $C > C_1$ (Figure 2).*

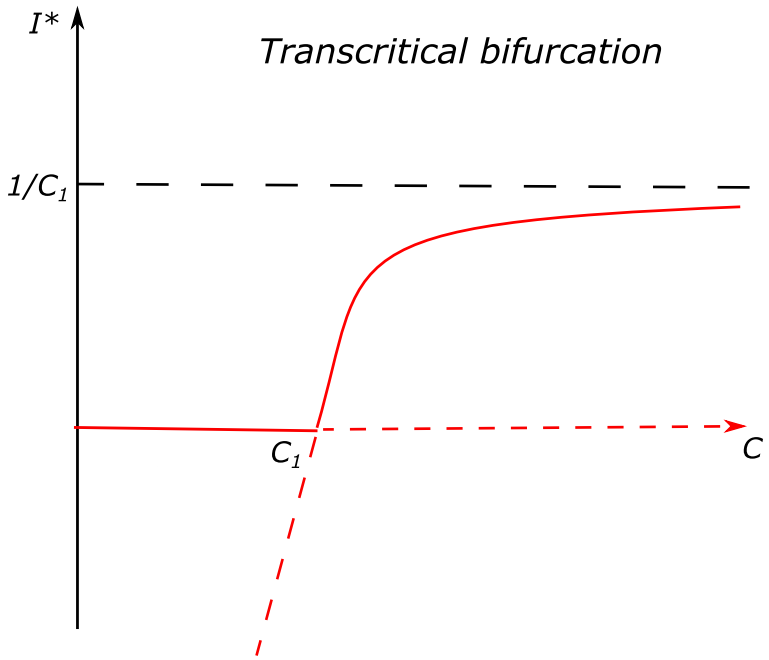

Figure 2: Bifurcation diagram for (3).

2.2. Global stability

In this subsection we investigate the global asymptotic stability of disease free equilibrium E_f and endemic equilibrium E^*.

We state the following theorem.

Theorem 2.4.

- *If* $R_0(\delta) \leq 1$, *then* $\widetilde{E_f}$ *is globally asymptotically stable in* $\widetilde{\Gamma}_D$.
- *If* $R_0(\delta) > 1$, *then* $\widetilde{E^*}$ *is globally asymptotically stable in* $\widetilde{\Gamma}_D \backslash \widetilde{\Lambda}_D$ *where* $\widetilde{\Lambda}_D = \{(S, 0, 0, R) \in \mathbb{R}^4_+ \backslash 0 \leq S + R \leq D\}$.

Proof. To study the global stability, we reduce system (3) as follow

$$\begin{cases} \dfrac{dS}{dt} = \alpha\delta D - \dfrac{CIS}{D} - \delta S, \\ \dfrac{dL}{dt} = \dfrac{CIS}{D} - (\sigma + \delta)L, \\ \dfrac{dI}{dt} = \sigma L - \delta I. \end{cases} \tag{6}$$

Since S, L and I are independent of R, we study the system (6) in the closed set

$$\widetilde{\Omega}_D = \{(S, L, I) \in \mathbb{R}^3_+ \backslash 0 \leq S + L + I \leq D\}.$$

- By means of a Lyapunov function we show that the disease free equilibrium $\widetilde{E_f}$ is globally asymptotically stable.

We set

$$V_1(S, L, I) = \frac{1}{2S^0}(S - S^0)^2 + L + \frac{\sigma + \delta}{\sigma} I + \frac{1}{2}(S - S^0 + L + I)^2.$$

$V_1(S, L, I) \geq 0$ and $V_1(S, L, I) = 0$ if and only if $S = S^0$ and $L = I = 0$.

Computing the time derivative of V_1, we find

$$\begin{aligned}
\frac{dV_1}{dt} &= \frac{1}{S^0}(S - S^0)\frac{dS}{dt} + \frac{dL}{dt} + \frac{\sigma + \delta}{\sigma}\frac{dI}{dt} + (S - S^0 + L + I)\left(\frac{dS}{dt} + \frac{dL}{dt} + \frac{dI}{dt}\right) \\
&= \frac{1}{S^0}(S - S^0)\left(\alpha\delta D - \frac{CIS}{D} - \delta S\right) + \frac{CIS}{D} - (\sigma + \delta) L \\
&\quad + \frac{\sigma + \delta}{\sigma}(\sigma L - \beta I) + (S - S^0 + L + I)(\alpha\delta D - \delta S - \delta L - \delta I) \\
&= \left(\frac{S - S^0}{S^0}\right)\left(-\frac{CIS}{D} - \delta(S - S^0)\right) + \frac{CIS}{D} - (\sigma + \delta) L \\
&\quad + \frac{\sigma + \delta}{\sigma}(\sigma L - \delta I) - \delta(S - S^0 + L + I)^2 \\
&= -\frac{\delta}{S^0}(S - S^0)^2 + \left(2S - \frac{S^2}{S^0} - S^0\right)\frac{C}{D} I + \left(S^0\frac{C}{D} - \delta\frac{\sigma + \delta}{\sigma}\right) I - \delta(S - S^0 + L + I)^2.
\end{aligned}$$

We can show that the coefficients of the term I in the last equality are negative. In fact, since $S^0 = \alpha D$ and $\mathscr{R}_0(\delta) = \frac{C\sigma\alpha}{\delta(\delta+\sigma)}$, we have $S^0\frac{C}{D} - \delta\frac{\sigma+\delta}{\sigma} = \frac{C\alpha(\mathscr{R}_0(\delta)-1)}{\mathscr{R}_0(\delta)}$, which is negative for $R_0(\delta) \leq 1$. Further, we have $2S - \frac{S^2}{S^0} - S^0 \leq 0$ for $S \geq 0$.

Thus, for $R_0(\delta) \leq 1$ we have $\frac{dV_1}{dt}(S, I, L) \leq 0$ for all $(S, L, I) \in \widetilde{\Omega}_D$. And $\frac{dV_1}{dt} = 0$ if and only if $(S, L, I) = (S^0, 0, 0)$ holds.

Thus, the only invariant set contained in $\widetilde{\Omega}_D$ is $\{\widetilde{E}_f\}$. Hence, LaSalle's theorem implies convergence of the solutions (S, L, I) to $(S^0, 0, 0)$ if initial values are in $\widetilde{\Omega}_D$.

Further, we have

$$\lim_{t\to+\infty} I(t) = 0 \Leftrightarrow \forall\epsilon > 0, \exists A_\epsilon > 0 \text{ such that for } t > A_\epsilon \text{ we have } |I(t)| < \epsilon.$$

Let $\epsilon > 0$, from equation four of system (3) we have

$$-(\epsilon\frac{C}{D} + \delta)R + (1 - \alpha)\delta D \leq \frac{d}{dt}R(t) \leq -(-\epsilon\frac{C}{D} + \delta)R + (1 - \alpha)\delta D.$$

After integrating between A_ϵ and t we obtain

$$\frac{(1-\alpha)\delta D}{\epsilon\frac{C}{D}+\delta} \leq \lim_{t\to\infty} R(t) \leq \frac{(1-\alpha)\delta D}{-\epsilon\frac{C}{D}+\delta}$$

for all ϵ such that $0 < \epsilon < \frac{D}{C}\delta$. Thus $\lim_{t\to\infty} R(t) = (1-\alpha)D = R^0$.

Therefore, for $\mathscr{R}_0(\delta) \leq 1$ the disease free equilibrium $\widetilde{E_f}$ is globally asymptotically stable in $\widetilde{\Gamma_D}$.

- Let

$$V_2(S,L,I) = \frac{\delta D}{CS^*}(S - S^* - S^* \ln\frac{S}{S^*}) + \frac{\delta D}{CS^*}(L - L^* - L^* \ln\frac{L}{L^*}) + (I - I^* - I^* \ln\frac{I}{I^*})$$

for $(S,L,I) \in \widetilde{\Gamma_D}\backslash\widetilde{\Lambda_D}$.

Then, we have

$$\begin{aligned}
\frac{dV_2}{dt} &= \frac{\delta D}{CS^*}(1-\frac{S^*}{S})\frac{dS}{dt} + \frac{\delta D}{CS^*}(1-\frac{L^*}{L})\frac{dL}{dt} + (1-\frac{I^*}{I})\frac{dI}{dt} \\
&= \frac{\delta D}{CS^*}(-\frac{\delta}{S}(S-S^*)^2 - \frac{CS^*}{DS}(I-I^*)(S-S^*) - \frac{C}{D}(S-S^*)I\frac{S-S^*}{S}) \\
&\quad + \frac{\delta D}{CS^*}(\frac{CIS}{D} - (\sigma+\delta)L - \frac{L^*}{L}\frac{CIS}{D} + L^*(\sigma+\delta)) \\
&\quad + (\sigma L - \delta I - \frac{I^*}{I}\sigma L + I^*\delta) \\
&= \frac{\delta D}{CS^*}(-\frac{\delta}{S}(S-S^*)^2 + \frac{CS^{*2}}{D}\frac{I}{S} - \frac{CS^{*2}I^*}{DS} - \frac{C}{D}IS - \frac{C}{D}S^{*2}\frac{I}{S} + 2\frac{C}{D}S^*I) \\
&\quad + \frac{\delta D}{CS^*}(\frac{C}{D}IS - (\sigma+\delta)L - \frac{CL^*}{D}\frac{IS}{L} + L^*(\sigma+\delta)) \\
&\quad + \sigma L - \delta I - \frac{I^*}{I}\sigma L + I^*\delta \\
&= -\frac{\delta D}{CS^*}\frac{\delta}{S}(S-S^*)^2 + \delta I^* - \frac{\delta S^*I^*}{S} - \frac{\delta}{S^*}IS + \delta I \\
&\quad + \frac{\delta}{S^*}IS - \frac{\delta D}{CS^*}(\sigma+\delta)L - \frac{\delta L^*}{S^*}\frac{IS}{L} + \frac{\delta D}{CS^*}L^*(\sigma+\delta) \\
&\quad + \sigma L - \delta I - \frac{I^*}{I}\sigma L + I^*\delta \\
&= -\frac{\delta D}{CS^*}\frac{\delta}{S}(S-S^*)^2 + 2\delta I^* - \frac{\delta S^*I^*}{S} + \left(\sigma - \frac{\delta D}{CS^*}(\sigma+\delta)\right)L \\
&\quad - \frac{\delta L^*}{S^*}\frac{IS}{L} + \frac{\delta D}{CS^*}L^*(\sigma+\delta) - \frac{I^*}{I}\sigma L.
\end{aligned}$$

Since $S^* = \frac{D\delta(\sigma+\delta)}{C\sigma}$ and $L^* = \frac{\delta}{\sigma}I^*$, we have

$$\begin{aligned}\frac{dV_2}{dt} &= -\frac{\sigma}{\sigma+\delta}\frac{\delta}{S}(S-S^*)^2 + 3\delta I^* - \frac{\delta S^* I^*}{S} - \frac{\delta L^*}{S^*}\frac{IS}{L} - \frac{I^*}{I}\sigma L \\ &= -\frac{\sigma}{\sigma+\delta}\frac{\delta}{S}(S-S^*)^2 - \delta I^*\left(\frac{S}{S^*}\frac{\delta}{\sigma}\frac{I}{L} + \frac{S^*}{S} + \frac{\sigma}{\delta}\frac{L}{I} - 3\right)\end{aligned}$$

which is negative if $(\frac{S}{S^*}\frac{\delta}{\sigma}\frac{I}{L} + \frac{S^*}{S} + \frac{\sigma}{\delta}\frac{L}{I} - 3)$ is positive.

Set $u = \frac{S^*}{S}$ and $v = \frac{\sigma}{\delta}\frac{L}{I}$ and consider the function

$$h(u,v) = u + v + \frac{1}{vu} - 3.$$

Computing the first derivatives of h, we have $\frac{\partial h(u,v)}{\partial u} = 1 - \frac{1}{u^2 v}$ and $\frac{\partial h(u,v)}{\partial v} = 1 - \frac{1}{uv^2}$. Moreover $\frac{\partial h(1,1)}{\partial u} = 0$ and $\frac{\partial h(1,1)}{\partial v} = 0$, thus $(1,1)$ is a critical point of h. For the second derivatives of h, we have $\frac{\partial^2 h(u,v)}{\partial u \partial v} = \frac{1}{u^2 v^2}$, $\frac{\partial^2 h(u,v)}{\partial u^2} = \frac{2}{vu^3}$ and $\frac{\partial^2 h(u,v)}{\partial v^2} = \frac{2}{uv^3}$.

We obtain

$$Hess(h)(1,1) = \left(\frac{\partial^2 h(1,1)}{\partial u \partial v}\right)^2 - \frac{\partial^2 h(1,1)}{\partial u^2}\frac{\partial^2 h(1,1)}{\partial v^2} = 3.$$

We have $Hess(h)(1,1) < 0$ and $\frac{\partial^2 h(1,1)}{\partial u^2} > 0$, therefore the function h has a minimum point $(u,v) = (1,1)$.

Moreover, $h(1,1) = 0$ and $h \geq 0$ which imply that $(\frac{S}{S^*}\frac{\delta}{\sigma}\frac{I}{L} + \frac{S^*}{S} + \frac{\sigma}{\delta}\frac{L}{I} - 3)$ is positive.

Thus $\frac{d}{dt}V_2(S,L,I) \leq 0$.

The function V_2 is a Lyapunov function for system (6), and $\frac{d}{dt}V_2(S,L,I) = 0$ if and only if $S = S^*$ and $\sigma L = \delta I$. Looking at the first equation in system (6) for $S = S^*$ we obtain $I = I^*$.

Hence $L = L^*$ and the equilibrium (S^*, L^*, I^*) is the only point satisfying $\frac{d}{dt}V_2(S,L,I) = 0$.

Thus the only invariant set contained in $\widetilde{\Omega}_D$ is $\{\widetilde{E^*}\}$, hence LaSalle's theorem implies convergence of the solutions (S,L,I) to (S^*, L^*, I^*) for all initial values in $\widetilde{\Omega}_D \backslash \widetilde{\Sigma}_D$ where

$$\widetilde{\Sigma}_D := \{(S,0,0) \in \mathbb{R}^3_+ \backslash 0 \leq S \leq D\}.$$

Moreover, using the procedure above, we can show that $\lim_{t\to\infty} R(t) = R^*$. Therefore, for $\mathscr{R}_0(\delta) > 1$ the endemic equilibrium $\widetilde{E^*}$ is globally asymptotically stable in $\widetilde{\Gamma}_D \backslash \widetilde{\Lambda}_D$.

□

Remark 2.1. If $\mathcal{R}_0(\delta) > 1$ and the initial conditions $(S(0), L(0), I(0), R(0)) \in \widetilde{\Gamma}_D$ with $(S(0), L(0), I(0)) \in \widetilde{\Sigma}_D$, then the solution of (3) converges to the disease free equilibrium $\widetilde{E}_f$.

Corollary 2.1.

- *If* $R_0(\delta) \leq 1$, *then* E_f *is globally asymptotically stable in* $\Gamma_D := \{(S, L, I, R, Q) \in \mathbb{R}^5 \backslash S + L + I + R + Q = D\}$.
- *If* $R_0(\delta) > 1$, *then* E^* *is globally asymptotically stable in* $\Gamma_D \backslash \Lambda_D$ *where* $\Lambda_D := \{(S, 0, 0, R, Q) \in \mathbb{R}^5_+ \backslash 0 \leq S + R + Q \leq D\}$. *In addition, if the initial conditions* $(S(0), L(0), I(0), R(0)) \in \Lambda_D$ *then the solution of (1) converges to the disease free equilibrium* E_f.

3. Non Constant Total Population (Case $\delta \neq \beta$)

Let us introduce new non-dimensional variables

$$s = \frac{S}{D},\ l = \frac{L}{D},\ i = \frac{I}{D},\ r = \frac{R}{D} \text{ and } q = \frac{Q}{D}.$$

We obtain the system

$$\begin{cases} s' = \alpha\beta - Cis - \beta s, \\ l' = Cis - (\sigma + \beta)l, \\ i' = \sigma l - \beta i, \\ r' = (1 - \alpha)\,\beta - Cir - \beta r, \\ q' = Cir - \beta q \end{cases} \tag{7}$$

with initial conditions

$$s(0) \geq 0,\ l(0) \geq 0,\ i(0) \geq 0,\ r(0) \geq 0,\ q(0) \geq 0. \tag{8}$$

From the homogeneity of (1), note that the total population size D does not appear in (7). Also observe that the first four equations in (7) do not depend on q and $s + l + i + r + q = 1$. Therefore the last equation can be omitted without loss of generality. Hence, system (7) can be reduced to

$$\begin{cases} s' = \alpha\beta - Cis - \beta s, \\ l' = Cis - (\sigma + \beta)l, \\ i' = \sigma l - \beta i, \\ r' = (1 - \alpha)\,\beta - Cir - \beta r. \end{cases} \tag{9}$$

We study (9) in the closed set

$$\widetilde{\Gamma}_1 = \{(s, l, i, r) \in \mathbb{R}^4_+ \backslash 0 \leq s + l + i + r \leq 1\}.$$

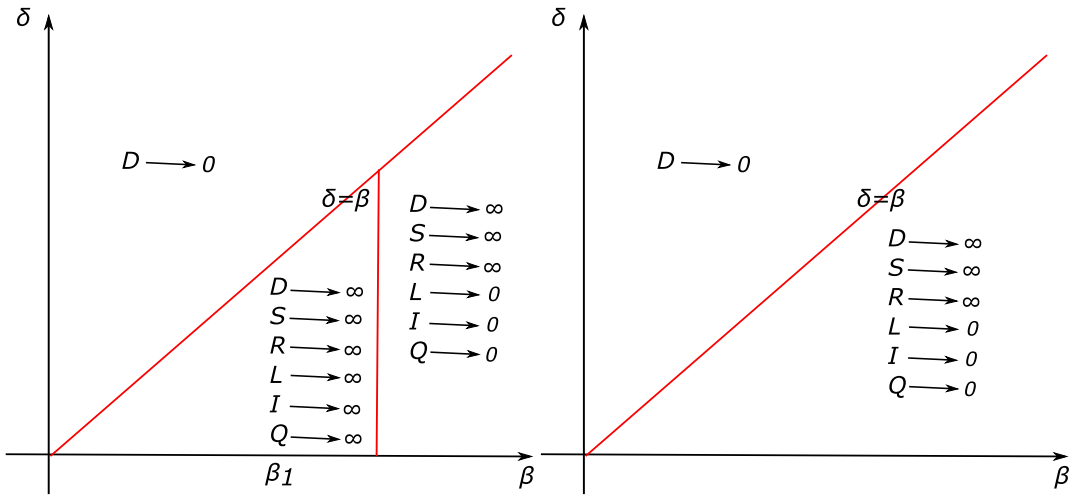

Figure 3: Bifurcation diagram for system (1). In left figure, the initial conditions $(S(0), L(0), I(0), R(0), Q(0)) \in \Gamma_{D_0} \backslash \Lambda_{D_0}$, whenever $\beta > \delta$ and $\beta > \beta_1 = \frac{-\sigma+\sqrt{\sigma^2+4C\sigma\alpha}}{2}$ the disease dies out while the total population increases exponentially. Although the disease spreads (that is, the number of infected grows in total number) when $\beta_1 < \beta$. In right figure, the initial conditions $(S(0), L(0), I(0), R(0), Q(0)) \in \Lambda_{D_0}$. Then, whenever $\beta > \delta$ the disease dies out while the total population increases exponentially. Note that when $\beta < \delta$, the total population decreases exponentially.

The set $\widetilde{\Gamma}_1$ is positively invariant with respect to (9).

Note that (9) is equivalent to (3) for $D = 1$ and $\delta = \beta$. The basic reproduction number is given by

$$\mathscr{R}_0(\beta) = \frac{C\alpha\sigma}{\beta(\sigma+\beta)}.$$

From Theorem 2.4 and Remark 2.1 we deduce the following results.

Theorem 3.1.

(i) *If* $\mathscr{R}_0(\beta) \leq 1$, *then* $e_f = (s^0, l^0, i^0, r^0) = (\alpha, 0, 0, 1-\alpha)$ *is the only equilibrium of (9), namely disease free equilibrium, it is globally asymptotically stable in* $\widetilde{\Gamma}_1$.

(ii) *If* $\mathscr{R}_0(\beta) > 1$, *then* e_f *is unstable and there exists a unique endemic equilibrium* $e^* = (s^*, l^*, i^*, r^*) = \left(\frac{\alpha}{\mathscr{R}_0}, \frac{\beta^2(\mathscr{R}_0-1)}{\sigma C}, \frac{\beta(\mathscr{R}_0-1)}{C}, \frac{(1-\alpha)}{\mathscr{R}_0}, \frac{(1-\alpha)(\mathscr{R}_0-1)}{\mathscr{R}_0}\right)$ *of (9), it is globally asymptotically stable in* $\widetilde{\Gamma}_1 \backslash \widetilde{\Lambda}_1$.

(iii) *If* $\mathscr{R}_0(\beta) > 1$, *then for all initial conditions* $(s(0), l(0), i(0), r(0)) \in \widetilde{\Lambda}_1$ *the solution of (9) converges to the disease free equilibrium* e_f.

From Theorem 3.1 we deduce the following results.

Corollary 3.1. *If* $\beta < \delta$, *then* $\lim_{t\to\infty} S(t) = \lim_{t\to\infty} L(t) = \lim_{t\to\infty} I(t) = \lim_{t\to\infty} R(t) = \lim_{t\to\infty} Q(t) = 0$.

Corollary 3.2. *Let* $\beta > \delta$.

(i) *If* $\mathscr{R}_0(\beta) \leq 1$, *then for all initial conditions* $(S(0), L(0), I(0), R(0), Q(0)) \in \Gamma_{D_0}$ *we have* $\lim_{t\to\infty} S(t) = \lim_{t\to\infty} R(t) = +\infty$ *and* $\lim_{t\to\infty} L(t) = \lim_{t\to\infty} I(t) = \lim_{t\to\infty} Q(t) = 0$.

Table 1: Parameter values for (1).

parameter	value (case $\beta = \delta$)	value (case $\beta < \delta$)	value (case $\beta > \delta$)
α	0.43	0.43	0.43
β	0.0011	0.0011	0.00167
δ	0.0011	0.00167	0.0011
σ	0.0050	0.0050	0.0050
D_0	50	50	50

(ii) *If* $\mathcal{R}_0(\beta) > 1$, *then for all initial conditions* $(S(0), L(0), I(0), R(0), Q(0)) \in \Gamma_{D_0} \backslash \Lambda_{D_0}$ *we have* $\lim_{t\to\infty} S(t) = \lim_{t\to\infty} L(t) = \lim_{t\to\infty} I(t) = \lim_{t\to\infty} R(t) = \lim_{t\to\infty} Q(t) = +\infty$.

(iii) *If* $\mathcal{R}_0(\beta) > 1$, *then for all initial conditions* $(S(0), L(0), I(0), R(0), Q(0)) \in \Lambda_{D_0}$ *we have* $\lim_{t\to\infty} S(t) = \lim_{t\to\infty} R(t) = +\infty$ *and* $\lim_{t\to\infty} L(t) = \lim_{t\to\infty} I(t) = \lim_{t\to\infty} Q(t) = 0$.

Remark 3.1. In the above results we can see that asymptotic behaviors of total population and infection subclasses depend on the values of birth and death rates and initial conditions (Figure 3).

4. Numerical Simulations

In this section we give numerical simulations for our model. The parameters values used were inspired from [4, 5] (see Table 1).

We discuss the simulation results of the system (1) according to the values of β and δ.

4.1. Case $\beta = \delta$

(1) When $\mathcal{R}_0(\delta) \leq 1$, the disease free equilibrium E_f is globally asymptotically stable in Γ_D (see Figure 4).

(2) When $\mathcal{R}_0(\delta) > 1$, the endemic equilibrium E^* is globally asymptotically stable in $\Gamma_D \backslash \Lambda_D$ and if the initial condition is in Λ_D, the solution tends to disease free equilibrium E_f (see Figure 5).

4.2. Case $\beta < \delta$

In this case we have extinction of total dog population (see Figure 6).

4.3. Case $\beta > \delta$

(1) When $\mathcal{R}_0(\beta) \leq 1$ (*i.e.* $\beta > \beta_1$), the disease dies that is $\lim_{t\to\infty} L(t) = I(t) = Q(t) = 0$, and total dog population increases exponentially, that is $\lim_{t\to\infty} N(t) = S(t) = R(t) = \infty$ (see Figure 7).

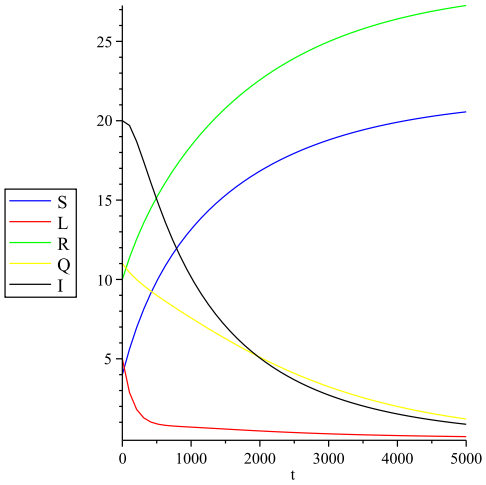

Figure 4: Simulation results for $\mathscr{R}_0(\delta) = 0.4806$, $C = 0.0015$ and initial condition $(S(0), L(0), I(0), R(0), Q(0)) = (4, 5, 20, 10, 11)$

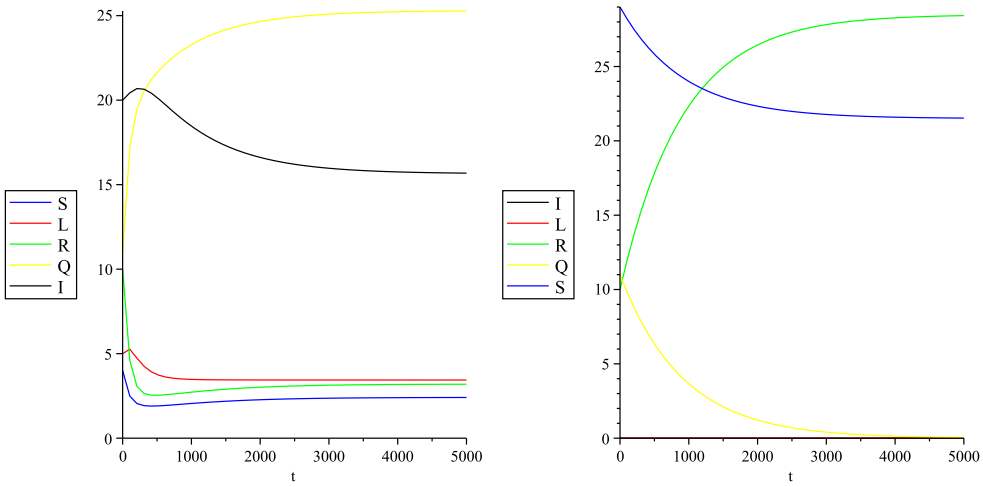

Figure 5: Simulation results for $\mathscr{R}_0(\delta) = 8.9076$, $C = 0.0278$ and initial condition $(S(0), L(0), I(0), R(0), Q(0)) = (4, 5, 20, 10, 11)$ (resp. $(S(0), L(0), I(0), R(0), Q(0)) = (29, 0, 0, 10, 11)$) in left (resp. right) figure.

(2) When $\mathscr{R}_0(\beta) > 1$ (*i.e.* $\beta < \beta_1$), all subclasses of D grow and we have $\lim_{t\to\infty} L(t) = I(t) = Q(t) = S(t) = R(t) = \infty$ if the initial condition is in $\Gamma_D \backslash \Lambda_D$. The disease dies if the initial condition is in Λ_D, in this case we have $\lim_{t\to\infty} L(t) = I(t) = Q(t) = 0$ and $\lim_{t\to\infty} N(t) = S(t) = R(t) = \infty$ (see Figure 8).

5. Conclusions

We have studied a ZVL mathematical model considered in [1, 4, 5] where only constant dog population is considered. In our work we have investigated the both cases where D is constant or not, our main results are given in Theorems 3, 4 and 6. .

In fact, in the case of constant dog population, we have analyzed the bifurcation of disease free equilibrium E_f to the endemic equilibrium E^* (Theorem 3), where stability is transmitted from E_f to E^*, the bifurcation studied with respect to the force of infection C, this is possible

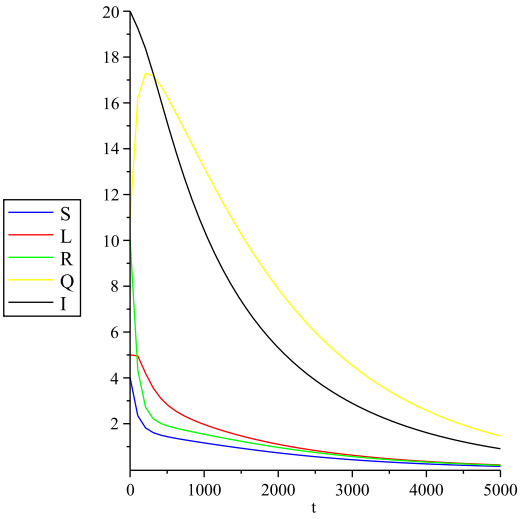

Figure 6: Simulation results for $\mathscr{R}_0(\beta) = 5.3659$, $C = 0.0278$ and initial condition $(S(0), L(0), I(0), R(0), Q(0)) = (4, 5, 20, 10, 11)$.

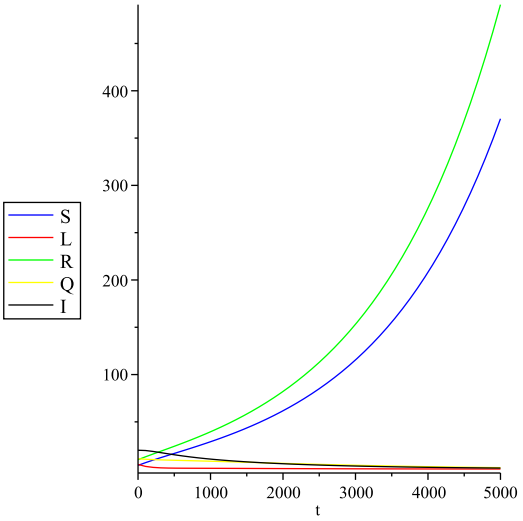

Figure 7: Simulation results for $\mathscr{R}_0(\beta) = 0.2895$, $C = 0.0015$ and initial condition $(S(0), L(0), I(0), R(0), Q(0)) = (4, 5, 20, 10, 11)$.

for $C = C_1$ corresponding to the basic reproduction number $\mathscr{R}_0(\delta) = 1$. In Theorem 4, we have found Lyapunov functions to prove the global stability of equilibria $\widetilde{E_f}$ and $\widetilde{E^*}$. These results allow us to find a global asymptotic stability domains of E_f and E^* (Corollary 5). In the case of non constant dog population, we have transformed (1) to a new model (7) equivalent to (1) by the mean of fraction of subclasses of the dog population D. We have obtained an interesting results concerning the behavior of each subclasses of D (Corollaries 7 and 8). Our results allows us to determine the cases where the infection goes to extinction or persistence depending on the parameter values. In fact, the birth and death rates values β and δ are important for the behavior of the size of total population D, which remains constant for equal values of birth and death rates. If $\beta < \delta$ the total population goes to zero (Figure 6). In the contrary, if $\beta > \delta$ then total population explodes, in this case the behavior of the infection depends on the value of the basic reproduction number $\mathscr{R}_0(\beta)$ and on domains of global asymptotic stability of the disease free

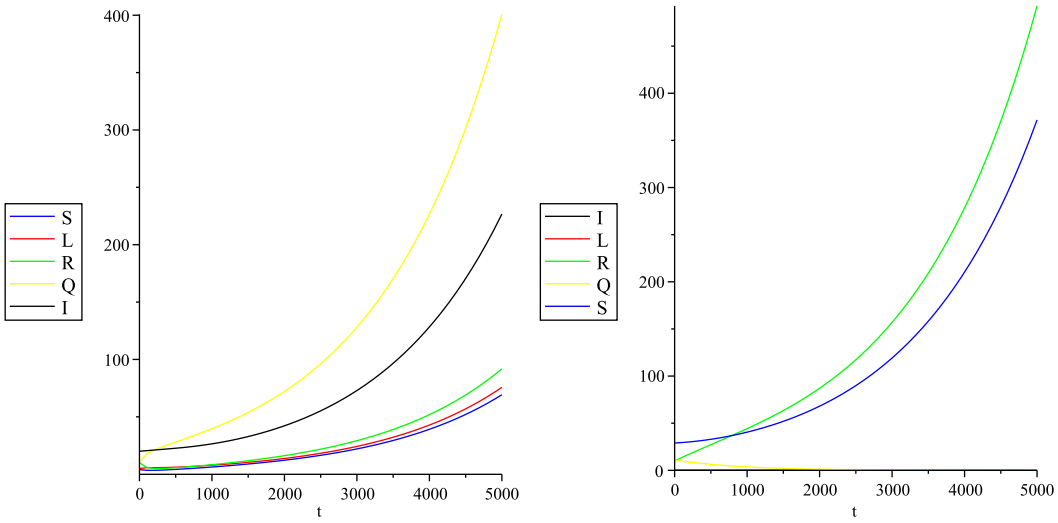

Figure 8: Simulation results for $\mathscr{R}_0(\delta) = 5.3659$, $C = 0.0278$ and initial condition $(S(0), L(0), I(0), R(0), Q(0)) = (4, 5, 20, 10, 11)$ (resp. $(S(0), L(0), I(0), R(0), Q(0)) = (29, 0, 0, 10, 11)$) in left (resp. right) figure.

and endemic equilibria. In fact, for $\mathscr{R}_0(\beta) \leq 1$ the infection subclasses go to extinction (Figure 7), but for $\mathscr{R}_0(\beta) > 1$ the behaviors of I and L depend on the domain of initial conditions, more precisely the infection vanishes only for $I(0) = L(0) = 0$ (Figure 8). If birth and death rates are equal then the behavior of infection subclasses goes to extinction for $\mathscr{R}_0(\delta) \leq 1$ (Figure 4). The infection persists if $\mathscr{R}_0(\beta) > 1$ for all initial conditions such that $I(0) \neq 0 \neq L(0)$ (Figure 5). Moreover, our results are illustrated by interesting numerical simulations for each cases.

This work could be continued, for example to see in what cases we must use control of the disease in order to reduce infection or to eradicate it. We can also consider the interaction with infected dog population and populations of sandflies and humans in order to obtain a more global model.

Competing Interests

The authors declare that they have no competing interests.

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